# Uniform asymptotic solutions of a system of two Schrodinger equations with potential-curve-crossing point



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# Uniform asymptotic solutions of a system of two Schrödinger equations with potential-curve-crossing point

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**Abstract.** A formal uniform asymptotic solution of the system of differential equations

$$h^2\frac{d^2U_1}{dz^2}+\Phi_1U_1=\alpha U_2 \qquad h^2\frac{d^2U_2}{dz^2}+\Phi_2U_2=\alpha U_1$$

for $z \in D$ and for real and small $h$ is obtained, when $D$ contains a curve-crossing point. Asymptotic approximations for the solutions are constructed in terms of parabolic cylinder functions. Analytical properties of the expansion coefficients are investigated.

## 1. Introduction

We consider the system of differential equations

$$\begin{aligned} h^2\frac{d^2U_1}{dz^2}+\Phi_1(z)U_1&=\alpha(z,\delta)U_2 \\ h^2\frac{d^2U_2}{dz^2}+\Phi_2(z)U_2&=\alpha(z,\delta)U_1 \end{aligned} \tag{1}$$

where $\Phi_1, \Phi_2, \alpha$ are functions of the complex variable $z$ in the domain $D$ which depend on the parameter $\delta$ in the complex domain $G$, $(0 \in G)$. We assume that $\alpha(z,0) \equiv 0$. The aim of this paper is to construct the uniform asymptotic expansion for the solution $U_1$ for small and positive values of the parameter $h$ in the domain $\Omega = D \times G$, which contains a crossing point $z = x_0$ (i.e. a point where $\Phi_1(x_0) = \Phi_2(x_0)$).

The problem of energy level crossing is of considerable practical importance. It appears in different branches of physics (see, for example, [1,2]), but it has no yet a general analytical solution. The set of equations (1) appears in physics in the problem of non-elastic collision of two atoms with masses $M_1$ and $M_2$ and is usually considered for $0 < z < \infty$, where $z$ is the distance between the atoms. The coefficients of the system have the form

$$\Phi_j = [\varepsilon - V_j]2M - \frac{h^2 l(l+1)}{r^2} \qquad \alpha = 2MV_{12}(z) \qquad M = \frac{M_1M_2}{M_1+M_2}$$

where $\varepsilon > 0$, $V_j$ are the energetic terms of the electron, and $V_{12}$ is the matrix element of the states of the electron. The WKB solutions of the set (1) are well known [3, 11]. They are not correct in the vicinity of the point $z_0$ where the energetic terms cross, i.e. when $V_1(z_0) = V_2(z_0)$. The connection formulae for the WKB solutions for $z < z_0$ and $z > z_0$ were obtained by Stueckelberg [3] with the help of phase-integral methods, and from the time-dependent point of view by Landau [4]. The theory has also been developed

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by analysis in the momentum representation by Bykovskii *et al* [6]. Later it was treated by Crothers *et al* [7, 8] but only the model problem with linear potentials $\Phi_1$, $\Phi_2$ and constant coupling function $\alpha$ was considered.

An analogous phenomenon of the change of one linear combination of WKB solutions into another in the vicinity of the crossing point of the potentials occurs in mechanics [5]. For example, it occurs in the system of two pendulums connected by a weightless spring $k$ whose length $a$ is equal to the distance between the points of suspension. The lengths of the pendulums $L_1$ and $L_2$ change adiabatically. Finally this system is described by the set of equations for small angles of inclination of the pendulums $\Theta_1$ and $\Theta_2$, namely

$$mL_1^2(t)\ddot{\Theta}_1 + (mgL_1(t) + ka^2)\Theta_1 = ka^2\Theta_2$$

$$mL_2^2(t)\ddot{\Theta}_2 + (mgL_2(t) + ka^2)\Theta_2 = ka^2\Theta_1.$$

The moment of time $t = t_0$ where $L_1(t_0) = L_2(t_0)$ corresponds to the crossing point of the potentials $\Phi_1$ and $\Phi_2$ for the system (1). In mechanics it is well known that in this case resonance occurs and the state of the system for $t < t_0$ and $t > t_0$ is described by different linear combinations of the solutions of the set of equations.

The asymptotic expansions of the solutions of the linear system (1) obtained in this paper are valid in the vicinity of the crossing point of the potentials and allows one to find out the solution of the system at any point, as well as the connection formulae for the WKB solutions.

In this paper we use the comparison equation technique [9, 10] and we establish uniform asymptotic approximations for the solutions of (1) in terms of parabolic cylinder functions. The important property for the fourth-order equation for the function $U_1$ is that both the coefficients of the equation and the turning points depend on $h$. In the case $\delta = 0$ ($\alpha \equiv 0$), the turning points do not coincide exactly. The distance between them is proportional to $h$. It has been shown in [10] how to construct the first term of the expansion for the solutions of the second-order differential equation with two close turning points and coefficients dependent on $h$. However, the case of a higher-order equation and arbitrary-order approximation has not been studied. Here one should mention an important contribution by Fedoruk [11] to the investigation of the problem under consideration.

We do not consider the asymptotic nature of the formal solution in the present paper, but the analytical properties of the coefficients in the asymptotic expansion are investigated.

## 2. WKB solutions

One can obtain from (1) the fourth-order differential equation for the function $U_1$:

$$U_1'''' + \frac{1}{h^2}U_1''(\Phi_1 + \Phi_2) + \frac{1}{h^4}U_1(\Phi_1\Phi_2 - \alpha^2) + 2\alpha\left(\frac{1}{\alpha}\right)' U_1''' + 2\alpha\left(\frac{\Phi_1}{\alpha}\right)' \frac{1}{h^2}U_1'$$
$$+\alpha\left(\frac{1}{\alpha}\right)'' U_1'' + \frac{1}{h^2}\alpha\left(\frac{\Phi_1}{\alpha}\right)'' U_1 = 0. \qquad (2)$$

The well known solutions of equation (2) when $D$ does not contain any turning points are [11]

$$
\begin{aligned}
y_{1,2}(z) &= \frac{\sqrt{\sqrt{E}-\Psi}}{\sqrt[4]{p_{10}^2 E}} \exp\left(\pm\frac{1}{h}\int^z p_{10}(t)\,\mathrm{d}t\right) \\
y_{3,4}(z) &= \frac{\sqrt{\sqrt{E}-\Psi}}{\sqrt[4]{p_{10}^2 E}} \exp\left(\pm\frac{1}{h}\int^z p_{30}(t)\,\mathrm{d}t\right).
\end{aligned}
\tag{3}
$$

In (3) we have introduced the notation $p_{i0}(z)$ $(i = 1, 2, 3, 4)$ for the roots of the equation

$$
l_0(z, p, \lambda) \equiv p^4 + (\Phi_1 + \Phi_2)p^2 + (\Phi_1\Phi_2 - \alpha^2) = 0 \tag{4}
$$

which can be written in the form

$$
\begin{aligned}
p_{10,20} &= \pm\sqrt{\Phi(z) + \sqrt{\Psi^2(z) + \alpha^2}} \\
p_{30,40} &= \pm\sqrt{\Phi(z) - \sqrt{\Psi^2(z) + \alpha^2}}
\end{aligned}
\tag{5}
$$

where

$$
\begin{aligned}
\Phi(z) &= -\tfrac{1}{2}[\Phi_1(z) + \Phi_2(z)] \\
\Psi(z) &= \tfrac{1}{2}[\Phi_1(z) - \Phi_2(z)] \\
E &= \Psi^2 + \alpha^2.
\end{aligned}
\tag{6}
$$

## 3. Reduction of equation (2)

It is easy to see that the WKB approximations of the functions $h^k U^{(k)}$ $(k = 0, 1, 2, \ldots)$ are of the same order in $h$. We shall see later that it is also true for the uniform approximations. That is why, keeping only higher-order terms in $h$ in equation (2), one can obtain the following equation:

$$
U_1'''' + \frac{1}{h^2}U_1''(\Phi_1 + \Phi_2) + \frac{1}{h^4}U_1(\Phi_1\Phi_2 - \alpha^2) = 0. \tag{7}
$$

We assume that the following conditions are satisfied:

(i) $\Phi_1$, $\Phi_2$ are the analytic functions for all $z \in D$, $\alpha$ is an analytic function for all $(z, \delta) \in \Omega = D \times G$.

(ii) At the point $x_0 \in D$ we have $\Phi_1(x_0) = \Phi_2(x_0) \neq 0$. Let us assume that $\Phi_1(x_0) > 0$.

(iii) $\alpha(x, 0) \equiv 0$ and $\alpha(x, \delta) \neq 0$ when $\delta \neq 0$ for any $x \in D$.

(iv) We suppose, that for any $\delta \in G$ the domain $D$ contains only two turning points $z_{10}$ and $z_{20}$, where $p_{10}(z_{i0}) = p_{30}(z_{i0})$ and $p_{20}(z_{i0}) = p_{40}(z_{i0})$ $(i = 1, 2)$. Here $p_{i0}$ $(i = 1, 2, 3, 4)$ are the roots of equation (7) and are given by (5).

To build the asymptotic expansion for the solutions of (7) we will follow the method suggested in [12]. For that purpose we transform the symbol $l_0$ (4) into the following form:

$$
l_0(p, h, \delta) = (p^2 + a_3 p + a_2)(p^2 + a_1 p + a_0) \tag{8}
$$

where

$$
\begin{aligned}
a_3 &= -(p_{10} + p_{30}) = -(-(\Phi_1 + \Phi_2) - 2\sqrt{\Phi_1\Phi_2 - \alpha^2})^{1/2} \\
a_2 &= p_{10}p_{30} = 2(-\Phi_1\Phi_2 - \alpha^2)^{1/2} \\
a_1 &= -a_3 \qquad a_0 = a_2.
\end{aligned}
$$

*Lemma 1.* The functions $a_i$ $(i = 1, \dots, 4)$ are analytic for all $(z, \delta) \in \Omega$.

Since equation (7) has no turning points other than $z_{10}$ and $z_{20}$, $\Phi_1\Phi_2 - \alpha^2 \neq 0$. It is easy also to see that $-(-(\Phi_1 + \Phi_2) - 2\sqrt{\Phi_1\Phi_2 - \alpha^2})^{1/2} \neq 0$ for all $z \in D$. Because of $\alpha^2 > 0$ $\Phi_1\Phi_2 > 0$, we have $\Phi_1\Phi_2 + \alpha^2 \neq 0$ for all $(z, \delta) \in \Omega$. We observe that $a_1$ and $a_2$ are the analytic functions of $z$, because of their being square roots of non-zero analytic functions.

We wish to obtain an asymptotic representations for four linearly independent solutions of (7) that are uniform in $D$ (including the points $z_1$, $z_2$). That is why we seek the asymptotic expansions of two linearly independent solutions of (7), corresponding to the first bracket of (8) in the form

$$U_{11,12}(z, \delta, h) = \exp\left(\frac{1}{2h}\int^z (p_{10} + p_{30})\,\mathrm{d}t\right)$$
$$\times\left(AU\left(\pm\frac{\mathrm{i}\tau_1}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_1\right) + \sqrt{h}BU\left(\pm\frac{\mathrm{i}\tau_1}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_1\right)\right) \qquad (9)$$

where $U(a, x)$ is the Weber function, which satisfies the equation $U'' - (\frac{x^2}{4} + a)U = 0$. We assume that $A$, $B$, $\tau\,\xi$ can be represented by

$$A(z, \delta, h) = \sum_{i=0}^{\infty} a_i(z, \delta)h^i \qquad (10)$$

$$B(z, \delta, h) = \sum_{i=0}^{\infty} b_i(z, \delta)h^i \qquad (11)$$

$$\tau(\delta, h) = \sum_{i=0}^{\infty} \tau_i(\delta)h^i \qquad (12)$$

$$\xi(z, \delta, h) = \xi(z, \delta). \qquad (13)$$

Using the anzatz (9) in (7) we get the following expressions for $\tau_0$ and $\xi$ [12]:

$$\tau_0(\lambda) = \frac{-1}{2\pi}\int_{z_1(\delta)}^{z_2(\delta)} \sqrt{F}\,\mathrm{d}t \qquad (14)$$

$$\int_{2i\sqrt{\tau_0(\delta)}}^{\xi(z,\delta)} \sqrt{-\xi^2/4 - \tau_0(\delta)}d\xi = \frac{1}{2}\int_{z_2(\delta)}^{z} \sqrt{F}\,\mathrm{d}t \qquad (15)$$

where $F = \frac{1}{4}(p_{10} - p_{30})^2$.

We choose the branches of the roots here in the following way: $\sqrt{F} \geqslant 0$ for$F \geqslant 0$.

*Lemma 2.* Equation (15) defines function $\xi(z, \delta)$ for all $z \in D$, $\delta \in G$ with the following properties:
(1) $\xi$ is analytic in $D \times G$;
(2) $\xi(\pm 2i\sqrt{\tau_0}) = z_{1,2}$;
(3) $\xi'(z, \delta) \neq 0$ for all $z \in D$, $\delta \in G$.

The proof of this lemma is based on Hartogs theorem (see [12])

To the next order of approximation for $a_0$ and $b_0$ we get the expresions [12]:

$$a_0(z) = k \exp\left(\int_{z_1}^{z} \Psi_1(t)\, \mathrm{d}t\right) \cosh\left(\int_{z_1}^{z} \left(\Psi_2(t) - \frac{\mathrm{i}}{2}\tau_1 \frac{\xi'}{\sqrt{\xi^2/4+\tau_0}}\right) \mathrm{d}t\right) \tag{16}$$

$$b_0(z) = \frac{k}{\sqrt{-\xi^2/4-\tau_0}} \exp\left(\int_{z_1}^{z} \Psi_1(t)\, \mathrm{d}t\right) \sinh\left(\int_{z_1}^{z} \left(\Psi_2(t) - \frac{\mathrm{i}}{2}\tau_1 \frac{\xi'}{\sqrt{\xi^2/4+\tau_0}}\, \mathrm{d}t\right)\right) \tag{17}$$

where

$$\Psi_1(z) = -\frac{\xi''}{\xi'} + a' \sum_{k=2,4} \frac{q_k}{q_k^2 - F} + \tfrac{1}{2}F' \sum_{k=2,4} \frac{1}{q_k^2 - F} \tag{18}$$

$$\Psi_2(z) = \frac{1}{2\sqrt{F}} \left(-a' + 2a' \sum_{k=2,4} \frac{F}{q_k^2 - F} + F' \sum_{k=2,4} \frac{q_k}{q_k^2 - F}\right) \tag{19}$$

$$a = \tfrac{1}{2}(p_{10} + p_{30}) \qquad q_k = p_k - a. \tag{20}$$

The parameter $\tau_1$ can be found from the condition that function $b_0(z)$ is analytic at $z = z_2$:

$$\tau_1 = -\frac{1}{\pi} \int_{z_1}^{z_2} \Psi_2(t)\, \mathrm{d}t. \tag{21}$$

As we have seen, the roots $p_{20}$, $p_{40}$ have branch points at the turning points $z_{10}$, $z_{20}$. But the functions $\Psi_1$ and$\Psi_2$ depend only on expressions $p_{20} + p_{40}$ and $p_{20}p_{40}$, which are analytic functions on$(z, \delta)$. This allows us to prove the following lemma.

*Lemma 3.* The coefficients $a_0$, $b_0$, defined by (16), (17) are analytic functions for all $(z, \delta) \in D \times G$.

The proof of the lemma in the case where the second bracket in $l_0(p, h, \delta)$ (8) does not have multiple zeros in $D$ has been carried out in [12] and it is easy to apply that proof to this case.

The other two linearly independent solutions of equation (7) can be found by changing the indices for the roots 1, 3 to 2, 4 in equations (14)–(20).

## 4. General equation

Now we shall consider the differential equation (2) with the coefficients and turning points depending on the small parameter $h$. The symbol of equation (2) has the form

$$l(z, p, \lambda) = p^4 + (\Phi_1 + \Phi_2)p^2 + (\Phi_1\Phi_2 - \alpha^2) + 2\alpha h \left(\frac{1}{\alpha}\right)' p^3 + 2\alpha h \left(\frac{\Phi_1}{\alpha}\right)' \frac{1}{h^2} p$$
$$+\alpha h^2 \left(\frac{1}{\alpha}\right)'' p^2 + \frac{1}{h^2}\alpha h^2 \left(\frac{\Phi_1}{\alpha}\right)''. \tag{22}$$

The roots of the characteristic equation $l(z, p, \delta) = 0$ and the turning points depend on the parameter $h$: $p_i = p_i(z, h)$ $(i = 1, 2, 3, 4)$; $z_{i,k}(h)$, $i = 1, 2$; $k = 1, 2$. Here $z_{i,1}(h)$ $(i = 1, 2)$ are the roots of the equation $p_1(z, h) = p_3(z, h)$ and $z_{i,2}(h)$ are the roots of the equation $p_2(z,h) = p_4(z,h)$. We accept that the functions $\Phi$ and $\Phi_2$ satisfy conditions (i), (ii) and (iii) of the previous section. Condition (iv) will be the following: for any $\delta \in G$ and $h < \varepsilon$ the domain $D$ contains only the turning points $z_{ik}(h,\delta)$, where $(i = 1,2;\ k = 1,2)$. In the previous part we used only the roots $p_i$ of the characteristic equation for constructing the asymptotical solution of equation (7). Now we can repeat all calculations and write the asymptotic solution of (2) in terms of $p_i(z,h)$.

However, we will not use this solution, because we cannot find the roots $p_i(z,h)$ of the fourth-order equation $l(z,p,\delta) = 0$ exactly. We either cannot expand the roots $p_i(z,h)$, because they have branch points at the turning points of equation (2). But the expressions $p_1(z,h) + p_3(z,h)$, $p_1(z,h)\,p_3(z,h)$, $p_2(z,h) + p_4(z,h)$ and $p_2(z,h)\,p_4(z,h)$ are analytic functions for all $(z,\delta) \in \Omega$, $h < \varepsilon$ as was the case for the roots of equation (7). This is shown in the following lemma.

*Lemma 4.* Let us consider the expression

$$l(p) = (p^2 + a_3 p + a_2)(p^2 + a_1 p + a_0) + c_3 p^3 + c_2 p^2 + c_1 p + c_0 \qquad (23)$$

where $a_i, c_i$, $i = 1,\ldots,4$ are analytic functions for all $(z;\delta) \in D \times G$, $c_i = \mathrm{O}(h)$, $c_i$ are analytic on $h$ for $h < \varepsilon$. Let $p_i \neq p_k$, $i = 1,2$, $k = 3,4$ for all $(z,\delta) \in D \times G$. Then:
*(i) The analytical functions $b_i, i = 1,2,3,4$ exist for all $(z,\delta) \in D \times G, h < \varepsilon$ such that*

$$l(p) = (p^2 + (a_3 + b_3)p + (a_2 + b_2))(p_2 + (a_1 + b_1 p + (a_0 + b_0)). \qquad (24)$$

(ii) If the vector $b$ is $b = (b_3, b_2, b_1, b_0)^t$, the coefficient $b_0$ in the expansion $b = b_0 + b_1 h + \cdots$ can be written in the form

$$b_0 = M^{-1} c \qquad (25)$$

where $M$ is the matrix

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ a_1 & a_3 & 1 & 1 \\ a_0 & a_2 & a_1 & a_3 \\ 0 & 0 & a_0 & a_2 \end{pmatrix}. \qquad (26)$$

*Proof of lemma 4.* Let $\beta$ be the vector $(0, b_3 b_1, b_3 b_0 + b_2 b_1, b_2 b_0)^t$. From (23) and (24) we get the equation

$$Mb + \beta = c. \qquad (27)$$

The determinant of the matrix $M$ is

$$\det M = (a_3 - a_2)(a_2 a_1 - a_3 a_0) - a_2 - a_0^2 = -(p_3 - p_1)(p_3 - p_2)(p_4 - p_1)(p_4 - p_2). \qquad (28)$$

It is clear that the $\det M = 0$ only in the case $p_i = p_k$, $i = 1,2$, $k = 3,4$. As follows from the formulation of the lemma, $\det M \neq 0$ and then we obtain from (27) the equation

$$b = M^{-1}\mathrm{c} - M^{-1}\beta. \qquad (29)$$

In accordance with the condition $c_i = \mathrm{O}(h)$, we have $\mathrm{c} = h\tilde{\mathrm{c}}$. Then $b = h\tilde{b}$ and $\beta = h^2\tilde{\beta}$. For small $h$: $(h < \varepsilon)$ we can solve the equation

$$\tilde{b} = M^{-1}\tilde{\mathrm{c}} - hM^{-1}\tilde{\beta} \qquad (30)$$

by successive approximations. The solution $\tilde{b}$ is the analytic function for all $(z,\delta) \in D \times G$ and $h < \varepsilon$, because of the function $c$ is analytic. The first-order term of the approximation on $h$ for $b$ can be found from (25).

In the case of curve-crossing we have the symbol of the equation

$$l(z,p,\delta) = (p^2 - (p_{10}+p_{30})p + p_{10}p_{30})(p^2 - (p_{20}+p_{40})p + p_{20}p_{40}) + c_3p^3 + c_1p + \cdots$$

$$= (p^2 - (p_1+p_3)p + p_1p_3)(p^2 - (p_2+p_4)p + p_2p_4) \tag{31}$$

where $c_3 = 2\alpha\left(\frac{1}{\alpha}\right)'$, $c_1 = 2\alpha\left(\frac{\Phi_1}{\alpha}\right)'$. Using the results of lemma 4 we see that the combinations of the roots $p_1+p_3$, $p_1p_3$, $p_2+p_4$ and $p_2p_4$ are analytic functions on $(z,\delta)$ and $h$. We can find two first terms in the series of them on $h$:

$$\begin{aligned}
p_1+p_3 &= p_{10}+p_{30} - \tfrac{1}{2}c_3 + \cdots \\
p_1p_3 &= p_{10}p_{30} + \frac{a_2}{2a_3}c_3 - \frac{1}{2a_3}c_1 + \cdots \\
p_2+p_4 &= p_{20}+p_{40} - \tfrac{1}{2}c_3 + \cdots \\
p_2p_4 &= p_{20}p_{40} + \frac{a_0}{2a_1}c_3 - \frac{1}{2a_1}c_1 + \cdots .
\end{aligned} \tag{32}$$

The functions $(p_1-p_3)^2$ and $(p_2-p_4)^2$ are analytic on $h$ and $z$, and we can find two first terms of the expansion

$$(p_1-p_3)^2 = f_1(t,h,\delta) + \mathrm{O}(h^2) \tag{33}$$

$$(p_2-p_4)^2 = f_2(t,h,\delta) + \mathrm{O}(h^2) \tag{34}$$

where

$$f_1(t,h,\delta) = (p_{10}-p_{30})^2 - \frac{2h}{p_{10}+p_{30}}\left(\alpha\left(\frac{1}{\alpha}\right)'(p_{10}^2+p_{30}^2) + 2\alpha\left(\frac{\Phi_1}{\alpha}\right)'\right) \tag{35}$$

$$f_2(t,h,\delta) = (p_{20}-p_{40})^2 - \frac{2h}{p_{20}+p_{40}}\left(\alpha\left(\frac{1}{\alpha}\right)'(p_{20}^2+p_{40}^2) + 2\alpha\left(\frac{\Phi_1}{\alpha}\right)'\right). \tag{36}$$

Since the right-hand side of (35), (36) has zeros at the turning points of equation (2), $p_1-p_3$ and $p_2-p_4$ are not analytic on $h$ and $z$. On substituting $p_1+p_3$, $(p_1-p_3)^2$, $p_2+p_4$, $(p_2-p_4)^2$, from (32)–(36) instead of $p_{10}+p_{30}$, $(p_{10}-p_{30})^2$, $p_{20}+p_{40}$, $(p_{20}-p_{40})^2$ in (9)–(21) we get the asymptotic solution of equation (2) as

$$\begin{aligned}
U_{11,12}(z) = \exp\left(\frac{1}{2h}\int^z \left(p_{10}+p_{30} - h\alpha\left(\frac{1}{\alpha}\right)'\right)\right)\,\mathrm{d}t \left(a_{01}U\left(\pm\frac{\mathrm{i}\tau_1}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_1\right)\right. \\
\left. + \sqrt{h}b_{01}U'\left(\pm\mathrm{i}\frac{\tau_1}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_1\right)\right)(1+\mathrm{O}(h))
\end{aligned} \tag{37}$$

$$\begin{aligned}
U_{13,14}(z) = \exp\left(\frac{1}{2h}\int^z \left(p_{20}+p_{40} - h\alpha\left(\frac{1}{\alpha}\right)'\right)\right)\,\mathrm{d}t \left(a_{02}U\left(\pm\frac{\mathrm{i}\tau_2}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_2\right)\right. \\
\left. + \sqrt{h}b_{02}U'\left(\pm\mathrm{i}\frac{\tau_2}{h}, \mathrm{e}^{\pm\frac{1}{4}\mathrm{i}\pi}\frac{1}{\sqrt{h}}\xi_2\right)\right)(1+\mathrm{O}(h)).
\end{aligned} \tag{38}$$

Here the parameters $\tau_i$ $(i = 1, 2)$ are defined by

$$\tau_i = \tau_{i0} + h\tau_{i1} + \mathrm{O}(h^2). \tag{39}$$

We can find the first term $\tau_{i0}$ from

$$\tau_{i0}(h,\delta) = -\frac{1}{2\pi}\int_{z_{i1}(\delta,h)}^{z_{i2}(\delta,h)} \sqrt{f_i(t,h,\delta)}\,\mathrm{d}t \tag{40}$$

and the function $\xi_i(z,\delta,h)$ is defined by

$$\int_{2\mathrm{i}\sqrt{\tau_{i0}(\delta,h)}}^{\xi_i(z,\delta,h)} \sqrt{-\xi^2/4 - \tau_{i0}(\delta,h)}\,\mathrm{d}\xi = \int_{z_{i2}(\delta,h)}^{z} \sqrt{f_i(t,h,\delta)}\,\mathrm{d}t. \tag{41}$$

The turning points $z_{i1}(h)$ and $z_{i2}(h)$ are the roots of the equation $f_i(t,h,\delta) = 0$. The coefficients $a_{0i}$ and $b_{0i}$ are given by

$$a_{i0}(z) = k\exp\left(\int_{z_{i1}}^{z} \Psi_{i1}(t)\,\mathrm{d}t\right)\cosh\left(\int_{z_{i1}}^{z}\left(\Psi_{i2}(t) - \frac{\mathrm{i}}{2}\tau_{i1}\frac{\xi_i'}{\sqrt{\xi_i^2/4+\tau_{i0}}}\right)\mathrm{d}t\right) \tag{42}$$

$$b_{i0}(z) = \frac{k}{\sqrt{-\xi_i^2/4-\tau_{i0}}}\exp\left(\int_{z_{i1}}^{z} \Psi_{i1}(t)\,\mathrm{d}t\right)\sinh\left(\int_{z_{i1}}^{z}\left(\Psi_{i2}(t) - \frac{\mathrm{i}}{2}\tau_{i1}\frac{\xi_i'}{\sqrt{\xi_i^2/4+\tau_{i0}}}\right)\mathrm{d}t\right) \tag{43}$$

where

$$\Psi_{i1}(z) = -\frac{\xi_i''}{\xi_i'} + a_i'\sum_{k=i+(-1)^{i+1},\,i+2+(-1)^{i-1}} \frac{q_{ik}}{q_{ik}^2 - F} + \tfrac{1}{2}F'\sum_{k=i+(-1)^{i+1},\,i+2+(-1)^{i-1}} \frac{1}{q_{ik}^2 - F} \tag{44}$$

$$\Psi_{i2}(z) = \frac{1}{2\sqrt{F}}\left(-a_i' + 2a_i'\sum_{k=i+(-1)^{i+1},\,i+2+(-1)^{i-1}} \frac{F}{q_{ik}^2 - F} + F'\sum_{k=i+(-1)^{i+1},\,i+2+(-1)^{i-1}} \frac{q_{ik}}{q_{ik}^2 - F}\right) \tag{45}$$

$$a_i = \tfrac{1}{2}(-1)^{i+1}(p_{10}+p_{30}) \qquad q_{ik} = p_k - a_i. \tag{46}$$

We can find the parameters $\tau_{i1}$ $(i = 1, 2)$ from the condition that function $b_{io}(z)$ is analytic at $z = z_{i2}$:

$$\tau_{i1} = -\frac{1}{\pi}\int_{z_{i1}}^{z_{i2}} \Psi_{i2}(t)\,\mathrm{d}t. \tag{47}$$

## 5. The case $\alpha \equiv 0$

Let us consider the asymptotic formulae (37), (38) for the particular case $\alpha \equiv 0$. We calculate the integrals (40), (47) by residues and get

$$\tau_{10} = -\frac{\mathrm{i}\Phi_1'(0)h}{\Phi_1'(0) - \Phi_2'(0)} + O(h^2) \tag{48}$$

$$\tau_{11} = \frac{\mathrm{i}}{2}\frac{\Phi_1'(0)+\Phi_2'(0)}{\Phi_1'(0)-\Phi_2'(0)} + O(h). \tag{49}$$

Finally for the parameter $\tau_1$ we have

$$\tau_1 = \tau_{10}(h) + h\tau_{11}(h) + \cdots = -\frac{\mathrm{i}}{2}h + \mathrm{O}(h^2). \tag{50}$$

Two Weber functions which we use in the expressions (37), (38) for $U_{11,12}$

$$U\left(\pm\tfrac{1}{2}, \exp(\pm\tfrac{1}{4}\mathrm{i}\pi)\frac{\xi_1}{\sqrt{h}}\right)$$

are linearly dependent. Consequently, the solutions $U_{11}$ and $U_{12}$ are linearly dependent. Consider the solution $U_{11}$. Then using the fact that in (37)

$$U\left(\pm\tfrac{1}{2}, \exp\left(\pm\tfrac{1}{4}\mathrm{i}\pi\right)\frac{\xi_1}{\sqrt{h}}\right) = \exp\left(-\frac{\mathrm{i}\xi_1^2}{4h}\right)$$

for $\xi_1 \geqslant 0$ we obtain

$$U_{11} \sim \frac{1}{\sqrt{p_{10}}}\exp\left(\frac{1}{h}\int^z p_{10}(t)\,\mathrm{d}t\right) \tag{51}$$

and for $\xi_1 \leqslant 0$

$$U_{11} \sim \frac{1}{\sqrt{p_{30}}}\exp\left(\frac{1}{h}\int^z p_{30}(t)\,\mathrm{d}t\right). \tag{52}$$

Finally, the solution $U_{11}$ for all real $z$ has the form

$$U_{11} \sim \frac{1}{\sqrt[4]{-\Phi_1}}\exp\left(\frac{1}{h}\int^z \sqrt{-\Phi_1(t)}\,\mathrm{d}t\right). \tag{53}$$

In the same way for the solution $U_{13}$ we get

$$U_{13} \sim \frac{1}{\sqrt[4]{-\Phi_1}}\exp\left(-\frac{1}{h}\int^z \sqrt{-\Phi_1(t)}\,\mathrm{d}t\right). \tag{54}$$

It is easy to see from (53), (54) that in the case $\alpha \equiv 0$ we get from uniform aproximations (37), (38) WKB solutions of the equation

$$h^2\frac{\mathrm{d}^2U_1}{\mathrm{d}z^2} + \Phi_1 U_1 = 0$$

which are valid for any $x \in D$.

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